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Tricritical scaling and renormalisation of ϕ^6 operators in scalar systems near four dimensions

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Abstract. Renormalisation of a field theory model for tricritical behaviour in which the operator ϕ^6 is essential for thermodynamic stability is discussed in $d = 4-\epsilon$ dimensions. The crossover scaling form of the equation of state is obtained explicitly to first order in ϵ for both positive and negative values of the four-spin coupling constant. Orthodox scaling, involving Gaussian tricritical exponents, is obtained, but is shown to be physically inappropriate in the ordered region of the symmetry plane. A reformulation of scaling, using classical tricritical exponents, is possible, but involves an additional parameter p with its own scaling exponent $\phi_p = -\frac{1}{2}(1-\epsilon)$. This parameter, introduced by Sarbach and Fisher for the many-component, spherical model limit, is related to the coefficient of $(\nabla\phi)^2$ in the field-theoretic Hamiltonian.

1. Introduction

Modern studies of critical phenomena have revealed the existence of a wide variety of multicritical points, at which one or more critical loci terminate. In general, each locus is characterised by its own set of critical exponents, while the exponents of the multicritical point are quite distinct. Furthermore, there corresponds to each set of critical exponents a borderline $d^\times$ of spatial dimensionality, above which the exponents are correctly given by mean field theory, but which may be different for the different types of critical behaviour. This last feature occurs, for example, in certain quantum-mechanical or finite-sized systems (Lawrie and Fisher 1978, Lawrie 1978a, b and references therein), at Lifshitz points, associated with the onset of helical ordering (Hornreich *et al* 1975), and at tricritical points (Riedel and Wegner 1973). Apart from a relatively small number of exactly soluble models, the most powerful tool presently available for the theoretical study of universal scaling behaviour near critical and multicritical points is the renormalisation group. An essential feature of this approach is the existence, for each type of asymptotic critical behaviour, of a fixed-point Hamiltonian, which contains only a few relevant operators. The relevance or irrelevance of a given operator is normally strongly dependent on dimensionality. Consequently, when the borderline dimensionalities for two types of critical behaviour are different, there may be operators, over and above those directly responsible for the crossover, which, while they are irrelevant for one fixed point, are relevant or 'dangerous' (in the sense of Fisher 1974a) for the other. The necessity of including such operators in the Hamiltonian can give rise to complications in the renormalisation group analysis. In particular, between the two borderlines, field-theoretic techniques involving standard

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dimensionality expansions afford an elegant and economical means of calculation. The dangerous operators then correspond to non-renormalisable interactions, and require special treatment.

In this work, we study the particular case of tricritical scaling in $d=4-\epsilon$ dimensions, which involves the renormalisation of ϕ^6 and related operators; however, the methods developed should be applicable to other situations of the type just described. (In earlier work (Lawrie 1978b) on quantal and finite-sized systems, however, the special nature of the crossover mechanism made it necessary to resort to less direct methods than those employed here.) As is well known, a simple mean field theory of tricritical points is obtained by minimising a free energy functional of the form

$$\mathscr{F} = -HM + \tfrac{1}{2}tM^2 + \tfrac{1}{4!}uM^4 + \tfrac{1}{6!}vM^6 \tag{1.1}$$

with respect to the magnetisation M. For the purposes of continuity with the usual renormalisation group approach to ordinary critical points, we suppose throughout this work that t is asymptotically linear in temperature, while u and v are asymptotically temperature-independent. In physical applications, for example to metamagnets (Griffiths 1973, Nelson and Fisher 1975) or to multicomponent fluids (Griffiths 1974, Fox 1978), different interpretations of the parameters appearing in (1.1) and correspondingly different assignments of critical exponents are normally appropriate. Moreover, we shall ignore the possibility of an additional term H_3M^3 which is required for a complete description of tricritical scaling. The minimisation condition yields an equation of state which may be written in the equivalent scaling forms

$$H = M^{\delta_t}\tilde{H}(tM^{-1/\beta_t}, uM^{-\phi_t/\beta_t}) = t^{\Delta_t}\bar{H}(Mt^{-\beta_t}, ut^{-\phi_t}), \tag{1.2}$$

where the set of *classical tricritical exponents* is given by

$$\beta_t = \tfrac{1}{4}, \qquad \delta_t = 5, \qquad \Delta_t = \beta_t\delta_t = \tfrac{5}{4}, \qquad \phi_t = \tfrac{1}{2}. \tag{1.3}$$

The variable v, which is strictly positive for thermodynamic stability, does not appear as a scaling variable in this description.

The location of the various critical lines is determined by solving the simultaneous equations

$$\partial H/\partial M = \partial^2 H/\partial M^2 = 0. \tag{1.4}$$

For $u > 0$, these equations are satisfied on the lambda line

$$t_\lambda = H_\lambda = M_\lambda = 0. \tag{1.5}$$

In this region, the equation of state may be written in an alternative form, namely

$$H = M^{\delta}\tilde{H}_\lambda(tM^{-1/\beta}, vM^{-\Psi/\beta}) = t^{\Delta}\bar{H}_\lambda(Mt^{-\beta}, vt^{-\Psi}), \tag{1.6}$$

where

$$\beta = \tfrac{1}{2}, \qquad \delta = 3, \qquad \Delta = \beta\delta = \tfrac{3}{2}, \qquad \Psi = -1 \tag{1.7}$$

are the classical exponents appropriate to the description of an ordinary critical point. On the other hand, for $u < 0$, one finds two critical loci,

$$t_c = 3u^2/2v, \tag{1.8}$$

$$H_c^{\pm} = \pm(4u^2/5v)(-6u/v)^{1/2} = (4u^2/5v)M_c^{\pm}, \tag{1.9}$$

which are the boundaries of the coexistence surfaces or 'wings' in the (H, t, u) phase diagram. In the vicinity of one of these loci, a scaling form analogous to (1.6) with the same exponents (1.7) and the substitutions

$$t \Rightarrow \tilde{t} = t - t_c, \qquad M \Rightarrow m = M - M_c, \qquad H \Rightarrow h = H - H_c - \tilde{t} M_c$$

is appropriate. The linear scaling field h measures, for fixed u and v, the deviation from the tangent $h = 0$ to the first-order surface at the critical point. In addition, an extra scaling variable

$$u_5 = vM_c \tag{1.10}$$

appears, with the crossover exponent

$$\Psi_5 = -\tfrac{1}{2}. \tag{1.11}$$

Since this exponent is negative, u_5 is actually an irrelevant variable, which could be ignored in the immediate vicinity of a wing critical point; it must, of course, be retained in order to recover the correct equation of state for positive u. The three critical loci meet at the tricritical point

$$H_t = t_t = M_t = u_t = 0. \tag{1.12}$$

We wish to investigate how this simple description is changed when the long-wavelength critical fluctuations, neglected in mean field theory, are taken into account by renormalisation group methods. Now, the borderline dimensionality for the critical loci is $d^\times = 4$, while it is believed that the tricritical point has $d^\times = 3$. Therefore one expects that, for $3 < d < 4$, the tricritical exponents will be correctly given by (1.3), while the critical exponents (1.7) will be changed. The latter can be calculated in the usual way as power series in $\epsilon = 4 - d$. It is known that tricritical behaviour is associated with the Gaussian fixed point of the renormalisation group (see e.g. Nelson and Fisher 1975). On this basis, several authors (Rudnick and Nelson 1976, Lawrie 1976, Bruce and Wallace 1976) have attempted to study the crossover from lambda-line to tricritical behaviour in $(4-\epsilon)$-dimensional, n-component models by setting $v = 0$, which is certainly permissible in the vicinity of the lambda line, and analysing the limit of the resulting theory as $u \to 0$. It is then found that the tricritical exponents are given, not by (1.3), but rather by those of the Gaussian model, namely

$$\beta_0 = \tfrac{1}{2}(1 - \epsilon/2), \qquad \Delta_0 = \tfrac{3}{2}(1 - \epsilon/6), \qquad \delta_0 = \Delta_0/\beta_0, \qquad \phi_0 = \epsilon/2, \tag{1.13}$$

which coincide with (1.3) only at the tricritical borderline $\epsilon = 1$. Apart from this discrepancy, these calculations have appeared to yield acceptable crossover scaling functions for various thermodynamic quantities, although the condition $v = 0$ clearly cannot be utilised in the region $u < 0$. We shall see, however, that this appearance is misleading. This point has been emphasised recently by Sarbach and Fisher (1978a, b) who have studied tricritical scaling in the spherical model limit $n \to \infty$. In this exactly soluble limit, they have been able to construct sensible scaling functions with the correct exponents (1.3), but at the expense of introducing an extra scaling variable associated with the range of interactions in the lattice model which is their starting point. We shall investigate the Ising-like case $n = 1$, thereby avoiding technical difficulties associated with transverse modes. In this case also, an extra scaling variable is required to yield sensible results, and indeed the procedure of Sarbach and Fisher is readily adapted here, although by a manoeuvre which appears somewhat *ad hoc* from the field-theoretic point of view.

The renormalisation of a field theory model corresponding to (1.1) is described in § 2. A scaling form for the equation of state, describing crossover from Ising-like to tricritical behaviour, is obtained for positive u in § 3 and for negative u in § 4. Finally, a self-contained summary of our principal results is given in § 5.

2. Renormalisation of a field theory model for tricriticality

Following the standard route, we define a field theory model corresponding to (1.1) by introducing the reduced Hamiltonian density (or Euclidean action density)

$$\mathscr{H}(x)=\tfrac{1}{2}(\nabla\phi_0(x))^2+\tfrac{1}{2}r_0\phi_0^2(x)+\tfrac{1}{4!}u_0\phi_0^4(x)+\tfrac{1}{6!}v_0\phi_0^6(x)+\Delta\mathscr{H}(x). \qquad (2.1)$$

The subscripts anticipate the need to define new, renormalised quantities in order to exhibit the scaling behaviour, while the notation t of (1.1) is reserved for the quantity related to r_0, which vanishes on the lambda line. Owing to the non-renormalisable nature of the ϕ_0^6 interaction, it is necessary to introduce operators (indeed an infinite number of them) other than those appearing explicitly in (1.1), in order to carry out a renormalisation programme in a consistent manner. The set of these operators and their coupling constants is denoted by $\Delta\mathscr{H}(x)$.

Renormalisation of (2.1) proceeds initially in the usual manner. One introduces a rescaled field

$$\phi(x)=Z_3^{-1/2}\phi_0(x) \qquad (2.2)$$

and renormalised parameters t, u, v. Their relation to the original quantities is determined by specifying the values of certain one-particle-irreducible vertex functions $\Gamma^{(n)}$, associated with expectation values of n distinct ϕ fields, at selected external momenta. The lambda line

$$r_0=r_{0c}(u_0, v_0) \qquad (2.3)$$

is determined by the vanishing of the inverse susceptibility

$$\chi^{-1}(r_0=r_{0c})=\Gamma^{(2)}(q^2=0;\ r_0=r_{0c})=0. \qquad (2.4)$$

On setting $r_0=r_{0c}$, the tricritical point

$$u_0=u_{0t}(v_0) \qquad (2.5)$$

is located by the simultaneous vanishing of the four-point function

$$\Gamma^{(4)}(q_i=0;\ r_0=r_{0c},\ u_0=u_{0t})=0. \qquad (2.6)$$

We shall take $u_0>u_{0t}$ throughout this section.

The object of further renormalisation is to absorb ultraviolet divergent contributions to the correlation functions into redefinitions of the field and coupling constant. Retaining the condition $r_0=r_{0c}$, and ignoring temporarily the contributions due to v_0, this is achieved by imposing the conditions

$$(\partial/\partial q^2)\Gamma^{(2)}(q^2)_{q^2=\mu^2}=1, \qquad (2.7)$$

$$\Gamma^{(4)}[p_i\cdot p_j=(\delta_{ij}-\tfrac{1}{4})\mu^2]=\mu^{\epsilon}u, \qquad (2.8)$$

which respectively determine the renormalisation coupling constant Z_3 and the renormalised coupling constant u. In these equations, μ denotes, as usual, an arbitrary non-zero parameter with the dimensions of momentum. The symmetric momentum point in (2.8) is chosen for technical convenience, so that the net momentum flowing through the one-loop diagram, figure 1(a) has magnitude μ, while the appearance of μ^{ϵ} on the right of (2.8) ensures that u is dimensionless.

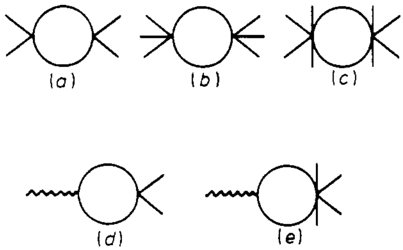

Figure 1. Graphs contributing to the renormalisation of (2.1). Wavy curves indicate insertions of the operator $\phi^2(x)$.

It now remains to renormalise the additional divergences due to the interaction $v_0\phi_0^6$. At first order of perturbation theory, the technique for doing this is standard (see e.g. Zimmerman 1971, Lowenstein 1970, Brezin *et al* 1976). It is that of renormalising correlation functions of the ϕ^4 theory, with single insertions of ϕ^6. This has been discussed by Amit (1978), while the entirely analogous renormalisation of ϕ^4 operators in $(6-\epsilon)$-dimensional, ϕ^3-dominated theories has been explained in detail by Amit *et al* (1977). For our case, the important result is that one must simultaneously renormalise all operators which, at $d=4$, have the same canonical dimension as ϕ^6, namely six or lower. The lower-dimensional operators have already been taken care of by the foregoing renormalisation: the required dimension-six operators are $(\nabla^2\phi)^2$ and $\phi^3\nabla^2\phi$. All other linearly independent operators are total derivatives of lower operators, which do not require further renormalisation. In statistical mechanical language, they contribute to the free energy only surface terms, which are negligible in the thermodynamic limit. Renormalisation of single insertions of these three operators is achieved by appropriate conditions on the corresponding vertex functions, namely

$$\Gamma^{(6)}[q_i\,.\,q_j=\tfrac{3}{4}(\delta_{ij}-\tfrac{1}{6})\mu^2]=\mu^{2\epsilon-2}v, \tag{2.9}$$

$$\sum_i \frac{\partial}{\partial q_i^2}\Gamma^{(4)}[q_i\,.\,q_j=(\delta_{ij}-\tfrac{1}{4})\mu^2]=\mu^{\epsilon-2}f, \tag{2.10}$$

$$(\partial/\partial q^2)^2\Gamma^{(2)}[q^2=\mu^2]=\mu^{-2}\Lambda^{-2}, \tag{2.11}$$

which define the dimensionless quantities v, f and Λ. The momentum point in (2.9) is again chosen so that the momentum flowing through the graph of figure 1(b) has magnitude μ. Differentiation of these three equations with respect to v, f and Λ^{-2} yields, in the limit where these quantities are zero, the 3×3 matrix of renormalisation constants required by the standard renormalisation theory of composite operators.

Even when this renormalisation has been carried out, higher-order terms of the perturbation series in v, f and Λ^{-2} contain additional divergences. At order v^2, for example, the eight-point vertex function contains the divergent contribution shown in figure 1(c). In order to subtract off this divergence, we must include in $\Delta\mathscr{H}$ a term $\frac{1}{8!}w_0\phi_0^8$. In general, this operator will itself require renormalisation. However, we are not interested in examining the corrections to scaling which directly involve such higher-order operators. It is therefore sufficient to set w_0 equal to just that function of the other parameters which subtract off the unwanted divergence. This can be arranged by the condition

$$\Gamma^{(8)}[p_i \cdot p_j = \tfrac{1}{2}(\delta_{ij} - \tfrac{1}{8})\mu^2] = 0, \tag{2.12}$$

and we now have

$$\Delta\mathcal{H} = \tfrac{1}{2}\Lambda_0^{-2}(\nabla^2\phi_0)^2 + \tfrac{1}{4!}f_0\phi_0^3\nabla^2\phi_0 + \tfrac{1}{8!}w_0\phi_0^8 + \ldots. \tag{2.13}$$

At higher orders in v, the addition of further higher-order operators to $\Delta\mathcal{H}$ will be necessary, and indeed an infinite number are required for a complete renormalisation of the theory. This is not disastrous, however. In order to calculate critical exponents correctly to order ϵ^n, one need renormalise only graphs containing up to n loops. At each finite order in the loopwise expansion, only a finite number of counterterms appear in $\Delta\mathcal{H}$. We shall carry out explicit calculations only to first order in ϵ. At this order, further simplifications occur, since, as shown by Amit (1978), there is in fact no mixing of the dimension-six operators. For the purposes of investigating tricritical behaviour, we need only the operator ϕ^6, and it is permissible to set $f = \Lambda^{-2} = 0$. When this is done, the conditions enumerated so far are sufficient to renormalise the massless ($r_0 = r_{0c}$) theory.

All that now remains is to determine the parameter t, by renormalising insertions of the operator ϕ^2. At one-loop order, two subtractions are required, corresponding to the graphs (d) and (e) in figure 1. Formally, this may be accomplished by adding to the Hamiltonian a term $\mu^2 t(x)\phi^2(x)$, where $t(x)$ is an external source, with Fourier transform $\hat{t}(q)$, and requiring

$$\lim_{\hat{t}\to 0} \frac{\delta}{\delta \hat{t}(q_3)} \Gamma^{(2)}(q_1, q_2)|_{q_1^2 = q_2^2 = q_3^2 = \mu^2} = 1, \tag{2.14}$$

$$\lim_{\hat{t}\to 0} \frac{\delta}{\delta \hat{t}(q_5)} \Gamma^{(4)}(q_1, q_2, q_3, q_4)|_{q_i \cdot q_j = \frac{5}{4}(\delta_{ij} - \frac{1}{5})\mu^2} = 0. \tag{2.15}$$

After renormalisation, we may set $t(x) = t$.

The relations between the renormalised and unrenormalised quantities are now completely determined to one-loop order. Each of the counterterms is proportional to the integral

$$\int \frac{d^d q}{q^2(\boldsymbol{q}+\boldsymbol{k})^2} = \frac{1}{\epsilon}\mu^{-\epsilon} S[1 + O(\epsilon)], \tag{2.16}$$

where $\boldsymbol{k}$ is a vector of magnitude μ, and the factor $S = 2\pi^{d/2}/(2\pi)^d\Gamma(d/2)$ arises, as usual, from angular integrations. We now have

$$\mu^{-2}(r_0 - r_{0c}) = t(1 + Su/2\epsilon) \tag{2.17}$$

$$\mu^{-\epsilon}(u_0 - u_{0t}) = u(1 + 3Su/2\epsilon) + Svt/2\epsilon \tag{2.18}$$

$$\mu^{2-2\epsilon} v_0 = v(1 + 15Su/2\epsilon) \tag{2.19}$$

$$\mu^{4-3\epsilon} w_0 = 35Sv^2/2\epsilon \tag{2.20}$$

$$Z_3 = 1. \tag{2.21}$$

Note that, owing to the final term in (2.18), u and v depend on r_0 as well as on u_0 and v_0. Thus the proportionality between $(r_0 - r_{0c})$ and t, which is exact when $v = 0$, now holds only asymptotically as $t \to 0$. Strictly speaking, (2.20) should also contain terms proportional to u^4 and $u^2 v$. However, these terms do not contain poles at $\epsilon = 0$, and thus can be neglected at this order.

The critical scaling properties of the correlation functions are contained in the renormalisation group equation

$$\left[\mu\frac{\partial}{\partial\mu} + W\frac{\partial}{\partial u} + \gamma_6\frac{\partial}{\partial v} - \nu^{-1} t\frac{\partial}{\partial t} - \frac{n}{2}\eta\right]\Gamma^{(n)} = 0 \tag{2.22}$$

obtained by applying the operator

$$\mu\frac{\partial}{\partial\mu}\bigg|_{r_0, u_0, v_0} = \mu\frac{\partial}{\partial\mu} + W\frac{\partial}{\partial u} + \gamma_6\frac{\partial}{\partial v} - \nu^{-1} t\frac{\partial}{\partial t} \tag{2.23}$$

to the relation

$$\Gamma_0^{(n)}(q_i; r_0, u_0, v_0) = Z_3^{-n/2}\Gamma^{(n)}(q_i; t, u, v, \mu). \tag{2.24}$$

In writing down these equations, we have assumed, incorrectly, that w_0 is a function only of r_0, u_0 and v_0. Actually, it is apparent from (2.12) that w_0, and hence $\Delta\mathcal{H}$, depends explicitly on the renormalisation point μ. In order to repair this mistake, one should treat w_0 as an independent variable, introducing a renormalised parameter w which would absorb the residual μ dependence. It would then be necessary to renormalise this parameter correctly. This in turn would necessitate the introduction of further higher-order operators, which would themselves require renormalisation, and so on. The solution of this problem is as follows. Let us suppose that the renormalisation has been carried out exactly. Equation (2.22) then involves additional parameters, w, w', ..., corresponding to all high-order operators. One may then solve (2.22) by the method of characteristics, introducing the auxiliary functions $\bar{w}$, $\bar{w}'$, ... Since we are not interested in the corrections to scaling involving the higher-order operators, we then set $\bar{w} = \bar{w}' = \ldots = 0$. The resulting correlation functions will be identical with those found by solving (2.22) as it stands. Elimination of the higher-order corrections amounts to imposing a set of conditions on the unrenormalised parameters appearing in the original Hamiltonian, which we are unable to write down explicitly. One may thus regard the differentiation (2.23) as being performed subject to these extra constraints, in which case (2.22) is correct.

The functions W, ν, γ_6 and η are determined by applying the operator (2.23) to (2.17), (2.18), (2.19), and (2.21). To order ϵ, we find

$$W(u) = -\epsilon u(1 - u/u^*), \tag{2.25}$$

$$\nu(u) = \tfrac{1}{2}(1 + \epsilon u/6u^*), \tag{2.26}$$

$$\gamma_6(u, v) = v(2 - 2\epsilon + 5\epsilon u/u^*), \tag{2.27}$$

$$\eta = \mu\frac{\partial}{\partial\mu}\ln Z_3|_0 = 0, \tag{2.28}$$

with $u^* = 2\epsilon/3S$, and it will be useful to define the additional functions

$$\beta(u) = \tfrac{1}{2}[2 - \epsilon + \eta]\nu(u) = \tfrac{1}{2}(1 - \epsilon/2 + \epsilon u/6u^*), \tag{2.29}$$

$$\Delta(u) = \tfrac{1}{2}(6 - \epsilon - \eta)\nu(u) = \tfrac{3}{2}(1 - \epsilon/6 + \epsilon u/6u^*), \tag{2.30}$$

$$\Psi(u) = -\nu(u)\gamma_6(u, v)/v = -(1 - \epsilon + 8\epsilon u/3u^*). \tag{2.31}$$

As usual, the critical exponents of the Ising-like fixed point $u = u^*$ are given by $\nu = \nu(u^*)$, $\beta = \beta(u^*)$, etc, while those of the Gaussian fixed point are $\nu_0 = \nu(0)$, $\beta_0 = \beta(0)$, etc. The exponents $\Psi = \Psi(u^*)$ and $\Psi_0 = \Psi(0)$ govern the corrections to scaling due to the operator ϕ^6, which will appear in terms of the scaling variable

$$vt^{-\Psi} = vt^{|\Psi|}. \tag{2.32}$$

In particular, near the Ising fixed point we have

$$\Psi = -[1 + \tfrac{5}{3}\epsilon + O(\epsilon^2)], \tag{2.33}$$

which agrees with the calculation of Amit (1978), although Amit expresses his result in a somewhat different form.

It should be remarked that a rather more complicated set of equations will be obtained at higher orders in ϵ. In the first place, it will be necessary to consider the mixing of the dimension-six operators. When the parameters $(v_1, v_2, v_3) = (v, f, \Lambda^{-2})$ are very small, the term $\gamma_6\, \partial/\partial v$ in (2.22) will be replaced by

$$\sum_{ij} \gamma_{ij}(u) v_i \frac{\partial}{\partial v_j}. \tag{2.34}$$

Diagonalisation of the matrix γ_{ij} will then guide the set of exponents Ψ_i governing the corrections due to these operators. In $(6-\epsilon)$-dimensional ϕ^3 theories, mixing of ϕ^4 and related operators already occurs at the one-loop level, and, in this context, some technical considerations concerning the diagonalisation of γ_{ij}, the identification of eigenoperators, and the consequences of equations of motion have been discussed by Amit *et al* (1977). These considerations are not important for the present calculation.

Secondly, the functions defined by (2.25)–(2.31) will themselves depend on the v_i, making the solution of the renormalisation group equation more complicated. They may also depend on t. At this order, the t-dependent term in (2.18) does not affect $W(u)$, owing to cancellation between the various derivatives in (2.23). A proof that such cancellations occur at all orders would be interesting, but we have not succeeded in producing one.

In order to discuss scaling behaviour near the tricritical point, we must add to the Hamiltonian (2.1) a magnetic field term $-H_0\phi_0(x)$. The corresponding renormalised dimensionless parameter H is given by

$$H_0\phi_0(x) = \mu^{3-\epsilon/2} H\phi(x); \qquad H_0 = \mu^{3-\epsilon/2} Z_3^{-1/2} H. \tag{2.35}$$

The part of the free energy depending directly on H is given in terms of the dimensionless magnetisation variable

$$M = \mu^{\epsilon/2-1}\langle\phi\rangle \tag{2.36}$$

by

$$\mu^{d} F(M) = \sum_n \frac{1}{n!}(\mu^{1-\epsilon/2} M)^n \Gamma^{(n)}(q_i = 0;\, u, v, t, \mu). \tag{2.37}$$

Since μ is the only dimensional parameter appearing in the renormalised theory, it may be factored out of the correlation functions when these are evaluated at zero momentum. It then appears on the right of (2.37) only as an overall factor μ^{d}, and $F(M)$ is itself dimensionless. Using (2.22), and multiplying throughout by $\nu(u)$, we obtain for the derivatives

$$F^{(n)}(M) = \partial^n F/\partial M^n \tag{2.38}$$

the renormalisation group equation

$$(t\,\partial/\partial t-\nu W\,\partial/\partial u+\Psi v\,\partial/\partial v+\beta M\,\partial/\partial M+n\beta-\mathrm{d}\nu)F^{(n)}=0. \tag{2.39}$$

In particular, the equation of state is given in scaling form by

$$H=F^{(1)}(M), \tag{2.40}$$

with

$$\beta(u)-\mathrm{d}\nu(u)=-\Delta(u), \tag{2.41}$$

while the inverse susceptibility is

$$\chi^{-1}=F^{(2)}(M), \tag{2.42}$$

with

$$2\beta(u)-\mathrm{d}\nu(u)=-\gamma(u). \tag{2.43}$$

These equations form the basis for the analysis in the next two sections.

3. Tricritical scaling: $u\geq 0$

In this section, we obtain to order ϵ a scaling form for the equation of state, which exhibits the crossover from lambda-line to tricritical behaviour. To this end, we integrate (2.39) with $n=1$ by the method of characteristics. This yields

$$H(u,v,t,M)=|t|^{(\Delta)}H(\bar{u},\bar{v},t/|t|,\bar{M}), \tag{3.1}$$

where the auxiliary functions $\bar{u}$, $\bar{v}$, and $\bar{M}$ are defined by

$$t\,\partial\bar{u}/\partial t=\nu(\bar{u})W(\bar{u}), \tag{3.2}$$

$$t\,\partial\bar{v}/\partial t=-\Psi(\bar{u})\bar{v}, \tag{3.3}$$

$$t\,\partial\bar{M}/\partial t=-\beta(\bar{u})\bar{M}, \tag{3.4}$$

with the boundary conditions $\bar{u}(t=1)=u$, $\bar{v}(t=1)=v$, $\bar{M}(t=1)=M$.

The prefactor in (3.1) is given by

$$|t|^{(\Delta)}=\exp\left(\int_1^{|t|}\frac{\mathrm{d}t'}{t'}\Delta(\bar{u}(t'))\right) \tag{3.5}$$

and has the limiting behaviour,

$$|t|^{(\Delta)}=|t|^{\Delta_0},\qquad \text{for } u=0, \tag{3.6}$$

$$|t|^{(\Delta)}=|t|^{\Delta},\qquad \text{for } u=u^*, \tag{3.7}$$

$$|t|^{(\Delta)}\approx|t|^{\Delta},\qquad \text{as } t\to 0 \text{ with } u>0, \tag{3.8}$$

At first order in ϵ, but only at this order, (3.2) may be solved explicitly for $\bar{u}$, yielding

$$\bar{u}=u^*z/(1+z), \tag{3.9}$$

with

$$z=\frac{(u/u^*)|t|^{-\phi_0}}{(1-u/u^*)},\qquad \phi_0=\tfrac{1}{2}\epsilon. \tag{3.10}$$

Crossover from Ising-like to Gaussian behaviour is conveniently expressed in terms of the function

$$X(|t|, u) = |t|^{\epsilon/2}\bar{u}/u = [(1+z)(1-u/u^*)]^{-1}, \tag{3.11}$$

which satisfies the differential equation

$$t\,\partial \ln X/\partial t = -\tfrac{1}{2}\epsilon|z|\partial \ln X/\partial z = \tfrac{1}{2}\epsilon\bar{u}/u^*, \tag{3.12}$$

and has the limiting forms

$$X(1, u) = X(|t|, 0) = 1, \tag{3.13}$$

$$X(|t|, u^*) = |t|^{\epsilon/2}, \tag{3.14}$$

$$X(|t|, u) \approx |t|^{\epsilon/2}, \qquad \text{as } |t| \to 0 \text{ with } u > 0. \tag{3.15}$$

The form (3.11) does not appear to be universal, owing to the residual dependence on u. However, X always appears in combination with other quantities, and the factor $1-u/u^*$ can always be absorbed into appropriate non-linear scaling fields, yielding scaling functions which are completely universal in form. Procedures for doing this have been indicated elsewhere (Lawrie 1977) and will not be discussed in detail here.

In terms of X, we have

$$|t|^{(\Delta)} = |t|^{3(1-\epsilon/6)/2}X^{1/2} = |t|^{\Delta_0}X^{2(\Delta-\Delta_0)/\epsilon}, \tag{3.16}$$

and, in analogous notation,

$$\bar{M} = M|t|^{-(1-\epsilon/2)/2}X^{-1/6} = M/|t|^{(\beta)} \tag{3.17}$$

$$\bar{v} = v|t|^{1-\epsilon}X^{16/3} = v/|t|^{(\Psi)}. \tag{3.18}$$

In order to exhibit the scaling form of the equation of state, one must now evaluate the free energy explicitly to one-loop order, and make the substitutions indicated in (3.1). Before doing this, it will be useful to have in hand an alternative formulation, using M rather than t as the basic scaling variable. On dividing (2.39) by $\beta(u)$, and following a procedure precisely analogous to the preceding one, we obtain

$$H(u, v, t, M) = |M|^{\delta}H(\tilde{u}, \tilde{v}, \tilde{t}, M/|M|) \tag{3.19}$$

where, to order ϵ, we have

$$\delta = \delta_0 = 3 + \epsilon. \tag{3.20}$$

Noting that, to order ϵ,

$$\phi_0/\beta_0 = \phi_0/\beta = \epsilon, \tag{3.21}$$

we obtain for the characteristic functions

$$\tilde{u} = \frac{u^*\tilde{z}}{1+\tilde{z}}, \qquad \tilde{z} = \frac{(u/u^*)|M|^{-\epsilon}}{(1-u/u^*)}, \tag{3.22}$$

$$\tilde{X}(|M|, u) = [(1+\tilde{z})(1-u/u^*)]^{-1}, \tag{3.23}$$

$$\tilde{t} = t|M|^{-2(1+\epsilon/2)}\tilde{X}^{1/3} = t/|M|^{(1/\beta)}, \tag{3.24}$$

$$\tilde{v} = v|M|^{2(1-\epsilon/2)}\tilde{X}^{15/3} = v/|M|^{(\Psi/\beta)}. \tag{3.25}$$

We now use (2.37) to calculate the free energy to one-loop order. One must sum the contributions from all graphs containing arbitrary numbers of insertions uM^2 and vM^4, arranged in a single loop. Taking into account the combinatorial factor associated with each such graph, we have

$$\frac{1}{2}\sum_{\substack{m,n\\ m+n\geq 1}}\left(-1^{m+n+1}\frac{(m+n-1)!}{m!n!}\tfrac{1}{2}uM^2\right)^m(\tfrac{1}{4!}vM^4)^n(q^2+t)^{-(m+n)}$$

$$=\tfrac{1}{2}\ln\left(1+\frac{\frac{1}{2}uM^2+\frac{1}{4!}vM^4}{q^2+t}\right) \tag{3.26}$$

which yields

$$F(M)=\tfrac{1}{2}tM^2+\tfrac{1}{4!}uM^4+vM^6+\tfrac{1}{2}\int \mathrm{d}^dq\ \ln[1+(\tfrac{1}{2}uM^2+\tfrac{1}{4!}vM^4)/(q^2+t)]-C(M), \tag{3.27}$$

where

$$C(M)=\tfrac{1}{2}(\tfrac{1}{2}uM^2+\tfrac{1}{4!}vM^4)\int\frac{\mathrm{d}^dq}{q^2}-\tfrac{1}{2}(\tfrac{1}{2}utM^2+\tfrac{1}{8}u^2M^4$$

$$+\tfrac{1}{24}vtM^4+\tfrac{1}{48}uvM^6+\tfrac{1}{1152}v^2M^8)\int\frac{\mathrm{d}^dq}{q^2(\boldsymbol{q}+\boldsymbol{k})^2} \tag{3.28}$$

denotes the contribution from counterterms subtracted in § 2, and $\hat{\boldsymbol{k}}$ is a unit vector. On performing the loop integrations, the equation of state may be written, to order ϵ, as

$$H=tM+\tfrac{1}{3!}uM^3+\tfrac{1}{5!}vM^5+(\epsilon/6u^*)(uM+\tfrac{1}{3!}vM^3)\tau\ln\tau \tag{3.29}$$

with

$$\tau=t+\tfrac{1}{2}uM^2+\tfrac{1}{4!}vM^4. \tag{3.30}$$

The scaling form is now obtained by making the substitutions (3.1) or (3.19). In the latter case, we obtain

$$H/M^\delta=\tilde{t}+\tfrac{1}{3!}\tilde{u}+\tfrac{1}{5!}\tilde{v}+(\epsilon/6u^*)(\tilde{u}+\tfrac{1}{3!}\tilde{v})\tilde{\tau}\ln\tilde{\tau}, \tag{3.31}$$

with $\tilde{\tau}$ defined in the analogous way, in terms of $\tilde{u}$ and $\tilde{v}$. As an illustration of the relation between the two formulations, let us set $u=u^*$ and $v=0$. In terms of the variables

$$y^2=u^*M^2|t|^{-2\beta},\qquad x=(u^*)^{-1/2\beta}t|M|^{-1/\beta} \tag{3.32}$$

we obtain, on exponentiating the logarithm,

$$(u^*)^{1/2}H=|t|^\Delta y[\tfrac{2}{3}+\tfrac{1}{3}(1+\tfrac{1}{2}y^2)^{\frac{1}{2}(\delta-1)}], \tag{3.33}$$

or

$$(u^*)^{-(1+\epsilon/2)}H=|M|^\delta[-\tfrac{1}{3}+(\tfrac{1}{2}+x)^{\Delta-\beta}]. \tag{3.34}$$

For small y, (3.33) is analytic, while the large-y behaviour reproduces the leading power in (3.34). Of course, (3.33) and (3.34) are identical only when the values of the exponents are substituted and the whole expressions expanded to order ϵ. Each form then reproduces (3.29) with $v=(u-u^*)=0$. When $v\neq 0$, corrections appear in terms of the variables $\tilde{v}/(u^*)^2$ or $\tilde{v}/(u^*)^2$, and one sees that the loopwise expansion yields scaling functions as power series in ϵ only if v is formally regarded as $O(\epsilon^2)$.

It is clear that, near the lambda line, (3.31) yields a correct scaling description, with corrections appearing in the form

$$(u-u^*)|M|^{-\phi_u/\beta}, \qquad v|M|^{-\Psi/\beta}, \tag{3.35}$$

with

$$\phi_u = -\tfrac{1}{2}\epsilon + O(\epsilon^2), \tag{3.36}$$

while the analogous expression obtained from (3.1) yields corrections varying with the appropriate powers of t. It might appear that our equations also describe directly the crossover to the tricritical point $u=0$, with tricriticality governed by the Gaussian exponents β_0, Δ_0, ϕ_0, etc, in which case our analysis would be complete. However, this appearance is easily seen to be misleading; indeed, if one sets $u=0$ in (3.31) to investigate the ordered region $t<0$ of the symmetry plane $H=0$, one finds

$$\tilde{t} + \tfrac{1}{5!}\tilde{v} + O(\tilde{v}^2) = 0, \tag{3.37}$$

or

$$t \approx -|M|^{1/\beta_0} v |M|^{-\Psi_0/\beta_0}/5! = -vM^4/5!. \tag{3.38}$$

Thus, on approaching the tricritical point in the ordered region, the relation between t and M is governed, not by the Gaussian exponent β_0 as suggested by (3.31), but rather by the classical tricritical exponent $\beta_t = \frac{1}{4}$, as in (1.2). In order to give a satisfactory account of the approach to tricriticality, we must therefore ask whether our equations can be rewritten in a form involving the classical tricritical exponents listed in (1.3) and in which, near the tricritical point, v appears as a constant rather than as a scaling variable. For notational convenience, avoiding the ratios ϕ/β and Ψ/β, we consider equation (3.1) which, to one-loop order, reads

$$\begin{aligned} H/|t|^{\frac{3}{2}(1-\epsilon/6)} = {} & M|t|^{-\frac{1}{2}(1-\epsilon/2)}X^{1/3} + \tfrac{1}{3!}\bar{u}M^3|t|^{-\frac{3}{2}(1-\epsilon/2)} + \tfrac{1}{5!}vM^5|t|^{-\frac{3}{2}(1-\epsilon/6)}X^{15/3} \\ & + (\epsilon/6u^*)(\bar{u}M|t|^{-\frac{1}{2}(1-\epsilon/2)}X^{1/3} + \tfrac{1}{3!}vM^3|t|^{-\frac{1}{2}(1+\epsilon/2)}X^{16/3})\bar{\tau}\ln\bar{\tau}, \end{aligned} \tag{3.39}$$

where $X(t,u)$ is still given by (3.11) and

$$\bar{\tau} = 1 + \tfrac{1}{2}\bar{u}M^2|t|^{-(1-\epsilon/2)}X^{-1/3} + \tfrac{1}{4!}vM^4|t|^{-1}X^{14/3}. \tag{3.40}$$

On multiplying (3.38) by $|t|^{\frac{3}{4}(1-\epsilon)}$, we obtain

$$\begin{aligned} H/|t|^{5/4} = {} & (M|t|^{-1/4})X^{1/3} + \tfrac{1}{3!}\bar{u}|t|^{-\frac{1}{2}(1-\epsilon)}(M|t|^{-1/4})^3 + \tfrac{1}{5!}v(M|t|^{-1/4})^5X^{15/3} \\ & + (\epsilon/6u^*)|t|^{\frac{1}{2}(1-\epsilon)}[\bar{u}|t|^{-\frac{1}{2}(1-\epsilon)}(M|t|^{-1/4})X^{1/3} \\ & + \tfrac{1}{3!}v(M|t|^{-1/4})^3X^{16/3}]\bar{\tau}\ln\bar{\tau}, \end{aligned} \tag{3.41}$$

with

$$\bar{\tau} = 1 + \tfrac{1}{2}\bar{u}|t|^{-\frac{1}{2}(1-\epsilon)}(M|t|^{-1/4})^2X^{-1/3} + \tfrac{1}{4!}v(M|t|^{-1/4})^4X^{14/3}. \tag{3.42}$$

We appear to have reached a dilemma! On the one hand, equations (3.31) and (3.39) obtained by direct solution of the renormalisation group equation have the formal appearance of scaling, but with physically inappropriate exponents. On the other hand, (3.41) embodies the correct tricritical exponents, but fails to scale factors of $|t|^{\frac{1}{2}(1-\epsilon)}$, which are equal to unity only on the tricritical borderline $\epsilon=1$. Essentially the same difficulty has been encountered recently by Sarbach and Fisher (1978a, b), who study tricriticality in a generalised spherical model, which is roughly equivalent to (2.1) in the many-component limit. Their resolution of the dilemma consists in having the foresight to include in their original formulation an additional variable p which, on being assigned the scaling exponent

$$\phi_p = -\tfrac{1}{2}(1-\epsilon), \tag{3.43}$$

precisely absorbs the residual dependence on t. (Actually, the exponent they define corresponds, in our notation, to $\phi_p/\phi_t = -(1-\epsilon)$.) The same parameter may be introduced into our equations by the following *ad hoc* manoeuvre. We modify the Hamiltonian (2.1) by writing

$$\mathscr{H}(x) = \tfrac{1}{2}p^{-2/d}[\nabla\phi_0(x)]^2 + \ldots. \tag{3.44}$$

From the field theory point of view, p is a spurious variable, since it can be removed by simple rescaling, provided that it is finite and non-zero. For this reason, it does not necessitate any further renormalisation. However, its introduction will permit us to retain the formal appearance of scaling, now with the correct classical tricritical exponents, while obtaining also scaling functions with a physically sensible behaviour in the ordered region of the symmetry plane. Moreover, one can give p a sound physical interpretation, if one imagines constructing (2.1) from an underlying lattice model in which the lattice spacing is denoted by a and the range of interactions by R_0. In that case, one has (Fisher 1974b, Sarbach and Fisher 1978a, b)

$$p \sim (a/R_0)^d. \tag{3.45}$$

In the limit of long-range interactions, $p \to 0$, spatial fluctuations in $\phi(x)$ are damped out, and one recovers the classical theory, as one should.

The net effect of the replacement (3.44) is to replace u^* in (2.25)–(2.31) and all subsequent equations by

$$u^* \Rightarrow u^*/p. \tag{3.46}$$

Using t as the basic scaling variable, we have

$$\bar{u}_t = |t|^{-\frac{1}{2}(1-\epsilon)}\bar{u} = u^* z_u/(1+z_u z_p), \tag{3.47}$$

$$X(|t|, u) = [(1+z_u z_p)(1-up/u^*)]^{-1}, \tag{3.48}$$

where, now, one has

$$z_u = \frac{(u/u^*)|t|^{-\phi_t}}{(1-up/u^*)}, \tag{3.49}$$

and

$$z_p = p|t|^{-\phi_p}, \tag{3.50}$$

while ϕ_p is given by (3.43). The equation of state, (3.41) with (3.42), now takes the scaling form

$$H/|t|^{\Delta_t} = \bar{M}_t X^{1/3} + \tfrac{1}{3!}\bar{u}_t\bar{M}_t^3 + \tfrac{1}{5!}v\bar{M}_t^5 X_t^5 + (\epsilon/6u^*)z_p(\bar{u}_t\bar{M}_t X^{1/3} + \tfrac{1}{3!}\bar{M}_t^3 X^{16/3})\bar{\tau}\ln\bar{\tau} \tag{3.51}$$

with

$$\bar{\tau} = 1 + \tfrac{1}{2}\bar{u}_t\bar{M}_t^2 X^{-1/3} + \tfrac{1}{4!}v\bar{M}_t^4 X^{14/3}, \tag{3.52}$$

and

$$\bar{M}_t = M|t|^{-\beta_t} = M|t|^{-1/4}. \tag{3.53}$$

These equations, together with an analogous modification of (3.31), constitute our final result for tricritical scaling with $u \geq 0$. In the limit $p = z_p = 0$, we have $X = 1$ and, as anticipated, the classical theory outlined in the Introduction is recovered. Since (3.39) has been modified only by the introduction of p, and by a redistribution of powers of t, the correct behaviour near the lambda line, which was manifest in (3.31) and (3.39), is automatically regained by setting $p = 1$ or, equivalently, by an appropriate rescaling of variables. In the disordered region of the symmetry plane, $H = 0$, $t > 0$, one may again eliminate p to reproduce the crossover to Gaussian tricritical behaviour found in previous analyses. At higher orders in ϵ, the scaling functions thus obtained will contain additional corrections to scaling involving v as an irrelevant variable, governed by the exponents Ψ and Ψ_0 of (2.31). Presumably these corrections are also contained in the work of Sarbach and Fisher, who do not, however, give them explicitly.

4. Tricritical scaling: $u \leq 0$

We now examine the scaling behaviour near the critical loci $[t, H] = [t_c(u, v), H_c^{\pm}(u, v)]$ with $u \leq 0$, and the crossover to tricritical behaviour as $u \to 0^-$. For brevity, we shall initially set $p = 1$; however, p may be reintroduced at a later stage via (3.45). As a first step, we locate t_c and M_c, using (3.29) and the conditions

$$\left.\frac{\partial H}{\partial M}\right|_c = \left.\frac{\partial^2 H}{\partial M^2}\right|_c = 0. \tag{4.1}$$

These conditions are satisfied when

$$t + \tfrac{1}{2}uM^2 + \tfrac{1}{4!}vM^4 = uM + \tfrac{1}{3!}vM^3 = 0, \tag{4.2}$$

which yields

$$t_c = 3u^2/2v, \tag{4.3}$$

$$M_c^{\pm} = \pm(-6u/v)^{1/2}, \tag{4.4}$$

while substitution in (3.29) gives

$$H_c^{\pm} = (4u^2/5v)M_c^{\pm}. \tag{4.5}$$

We see that these results are formally identical with (1.8) and (1.9), obtained for the classical theory, although, of course, u and v are now renormalised parameters. It is not clear whether this attractive feature will persist at higher orders.

In order to find the fixed point governing one of the wing critical lines, we make the change of variable

$$\phi(x) \Rightarrow \phi(x) + M_c \tag{4.6}$$

with $M_c = M_c^+$, say, and introduce the new temperature parameter

$$\dot{t} = t - t_c(u, v). \tag{4.7}$$

For $u > 0$, the renormalisation scheme of § 2 was carried out in the symmetry plane $H = 0$, while correlation functions valid for arbitrary values of the magnetic field could be constructed from those of the symmetric theory via (2.37). In the present case, the analogous procedure is to eliminate the term linear in $\phi(x)$ introduced by the shift (4.6).

Thus we add to the Hamiltonian a term $-(H_c+\dot{\imath}M_c)\phi(x)$, with $H_c=H_c^+$. That is, we restrict ourselves temporarily to the surface tangential to the wing at the critical line, by setting the linear scaling field

$$h=H-H_c-\dot{\imath}M_c, \tag{4.8}$$

introduced in § 1, equal to zero.

Apart from an unimportant constant, the Hamiltonian now reads

$$\mathscr{H}(x)=\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}\dot{r}_0\phi^2+\tfrac{1}{3!}\dot{H}_{30}\phi^3+\tfrac{1}{4!}\dot{u}_0\phi^4+\tfrac{1}{5!}\dot{u}_{50}\phi^5+\tfrac{1}{6!}\dot{v}_0\phi^6+\tfrac{1}{7!}\dot{u}_{70}\phi^7+\tfrac{1}{8!}\dot{w}_0\phi^8, \tag{4.9}$$

with

$$\dot{r}_0=r_{0c}+\tfrac{1}{2}u_{0t}M_c^2+\mu^2\dot{\imath}(1-Su/\epsilon), \tag{4.10}$$

$$\dot{H}_{30}=u_{0t}M_c+\tfrac{1}{2}Sv\dot{\imath}M_c/\epsilon, \tag{4.11}$$

$$\dot{u}_0=u_{0t}-2\mu^{\epsilon}u(1-3Su/\epsilon), \tag{4.12}$$

$$\dot{u}_{50}=\mu^{3\epsilon/2-1}vM_c(1-10Su/\epsilon), \tag{4.13}$$

$$\dot{v}_0=\mu^{2\epsilon-2}v(1-45Su/\epsilon), \tag{4.14}$$

$$\dot{u}_{70}=35\mu^{5\epsilon/2-3}Sv^2M_c/2\epsilon, \tag{4.15}$$

$$\dot{w}_0=35\mu^{3\epsilon-4}Sv^2/2\epsilon=w_0. \tag{4.16}$$

Note that $\dot{H}_{30}$ consists of just those counterterms required to ensure that

$$\dot{\Gamma}^{(3)}(q_i=0;\dot{\imath}=0)=0, \tag{4.17}$$

and that insertions of $\dot{\imath}\phi^2$ in the same vertex function are correctly renormalised. Since the changes of variables (4.6) and (4.7) do not involve any divergent quantities, the theory defined by (4.9)–(4.15) should already be correctly renormalised. Indeed, on introducing

$$\dot{u}=-2u, \tag{4.18}$$

$$\dot{u}_5=vM_c, \tag{4.19}$$

and, for consistency,

$$\dot{v}=v, \tag{4.20}$$

the relations (4.10) and (4.12) assume the same forms as (2.17) and (2.18) respectively, while (4.13)–(4.16) may be rewritten as

$$\dot{u}_{50}=\mu^{3\epsilon/2-1}\dot{u}_5(1+5S\dot{u}/\epsilon), \tag{4.21}$$

$$\dot{v}_0=\mu^{2\epsilon-2}[\dot{v}(1+15S\dot{u}/2\epsilon)+5S\dot{u}_5^2/\epsilon], \tag{4.22}$$

$$\dot{u}_{70}=35\mu^{5\epsilon/2-1}S\dot{u}_5\dot{v}/2\epsilon, \tag{4.23}$$

$$\dot{w}_0=35\mu^{3\epsilon-4}\dot{v}^2/2\epsilon. \tag{4.24}$$

One may check explicitly that precisely these relations are obtained by renormalising (4.9) along the lines of § 2.

The free energy deviation

$$\dot{F}(m)=\dot{F}(M-M_c)=F(M)-F(M_c) \tag{4.25}$$

and its derivatives satisfy the renormalisation group equation

$$\left(\dot{t}\frac{\partial}{\partial \dot{t}} - \nu(\dot{u})W(\dot{u})\frac{\partial}{\partial \dot{u}} + \Psi_5(\dot{u})\dot{u}_5\frac{\partial}{\partial \dot{u}_5} - \nu(\dot{u})\dot{\gamma}_6(\dot{u}, \dot{u}_5, \dot{v})\frac{\partial}{\partial \dot{v}} + \beta(\dot{u})m\frac{\partial}{\partial m} + n\beta(\dot{u}) - \mathrm{d}\nu(\dot{u})\right)\dot{F}^{(n)} = 0, \tag{4.26}$$

where $\nu(\dot{u})$, $W(\dot{u})$, and $\beta(\dot{u})$ are the functions defined in § 2, while the remaining coefficients are given by

$$\nu\dot{\gamma}_6 = (1-\epsilon+8\epsilon\dot{u}/3u^*)\dot{v} + 5\epsilon\dot{u}_5^2/3u^* = -\Psi(\dot{u})\dot{v} + 5\epsilon\dot{u}_5^2/3u^*, \tag{4.27}$$

$$\Psi_5 = -\tfrac{1}{2}(1-\tfrac{3}{2}\epsilon+7\epsilon\dot{u}/2u^*). \tag{4.28}$$

The exponent Ψ_5 governing corrections to scaling due to the operator ϕ^5 near the Ising-like fixed point $\dot{u} = u^*$ associated with the wing critical line is given by (4.28) as

$$\Psi_5 = -\tfrac{1}{2}(1+2\epsilon+\mathrm{O}(\epsilon^2)). \tag{4.29}$$

It should be remarked that (4.25) is not merely a trivial rewriting of (2.39), since the differentiation is now subject to the extra constraint $\dot{u}_{50}$ = constant, which gives rise to the additional corrections to scaling with the exponent Ψ_5. Also, (4.25) does not contain corrections associated with $\dot{H}_{30}$: the operator ϕ^3 is redundant, in the sense of Wegner (1974).

Solution of (4.25) follows exactly the path of § 3. In terms of the linear scaling field h defined in (4.8), we have for the equation of state

$$h(\dot{u}, \dot{u}_5, \dot{v}, \dot{t}, m) = |\dot{t}|^{(\Delta)} h(\hat{u}, \hat{u}_5, \hat{v}, \dot{t}/|\dot{t}|, \bar{m}). \tag{4.30}$$

Scaling of $\dot{u}$ and m is exactly as before, and we have

$$\hat{u} = u^*\dot{z}/(1+\dot{z}), \tag{4.31}$$

$$\dot{X}(|\dot{t}|, \dot{u}) = [(1+\dot{z})(1-\dot{u}/u^*)]^{-1}, \tag{4.32}$$

$$\bar{m} = m|\dot{t}|^{-(1-\epsilon/2)/2}\dot{X}^{-1/6} = m/|\dot{t}|^{(\beta)}, \tag{4.33}$$

$$|\dot{t}|^{(\Delta)} = |\dot{t}|^{3(1-\epsilon/6)/2}\dot{X}^{1/2}, \tag{4.34}$$

with

$$\dot{z} = \frac{(\dot{u}/u^*)|\dot{t}|^{-\epsilon/2}}{(1-\dot{u}/u^*)}. \tag{4.35}$$

The form of $\hat{v}$ is modified by the second term in (4.27), and the analogue of (3.18) becomes

$$\hat{v} = \dot{v}/|\dot{t}|^{(\Psi)} + 5\hat{u}_5^2(\dot{X}^{-2/3}-1)/\hat{u}, \tag{4.36}$$

while $\hat{u}_5$ is given by

$$\hat{u}_5 = \dot{u}_5|\dot{t}|^{(1-3\epsilon/2)/2}\dot{X}^{7/2} = \dot{u}_5/|\dot{t}|^{(\Psi_5)}. \tag{4.37}$$

In order to write the equation of state in scaling form with the correct tricritical exponents, it is necessary once again to introduce the variable p via (3.45). In analogy with (3.46)–(3.49), this yields

$$\hat{u}_t = |\dot{t}|^{-(1-\epsilon)/2}\hat{u} = u^*\dot{z}_u/(1+\dot{z}_u\dot{z}_p) \tag{4.38}$$

$$\dot{X} = [(1+\dot{z}_u\dot{z}_p)(1-\dot{u}p/u^*)]^{-1}, \tag{4.39}$$

with

$$\dot{z}_u = (\dot{u}/u^*)|t|^{-\phi_t}/(1-\dot{u}p/u^*), \tag{4.40}$$

$$\dot{z}_p = p|\dot{t}|^{-\phi_p}, \tag{4.41}$$

and ϕ_p and ϕ_t defined respectively by (3.42) and (3.50). Corrections due to u_5 now appear in terms of the scaling variable

$$\dot{z}_{5t} = \dot{u}_5|\dot{t}|^{-\Psi_{5t}}, \tag{4.42}$$

where, as expected, the exponent

$$\Psi_{5t} = \tfrac{1}{4} \tag{4.43}$$

agrees with the corresponding Gaussian exponent (cf (4.27))

$$\Psi_{50} = -\tfrac{1}{2}(1-\tfrac{3}{2}\epsilon) \tag{4.44}$$

only on the tricritical borderline $\epsilon = 1$. Since the exponent Ψ_{5t} is positive, we see that the operator ϕ^5 is actually relevant at the tricritical point. This is to be expected, since its canonical dimension is lower than that of ϕ^6. Of course, its coupling constant $\dot{u}_5 = vM_c$ vanishes at the tricritical point, as it must. It is also convenient to introduce the quantity

$$\dot{v}_t = \dot{v} + 5z_{5t}^2\dot{X}^{5/3}(\dot{X}^{-2/3}-1)/\dot{u}_t = |\dot{t}|^{\epsilon-1}\dot{X}^{-16/3}\dot{v}. \tag{4.45}$$

The equation of state then reads

$$\begin{aligned} h/|\dot{t}|^{\Delta_t} = {} & \bar{m}_t\dot{X}^{1/3} + \tfrac{1}{3!}\dot{u}_t\bar{m}_t^3 + \tfrac{1}{4!}\dot{z}_{5t}\bar{m}_t^4\dot{X}^{10/3} + \tfrac{1}{5!}\dot{v}_t\bar{m}_t^5\dot{X}^5 \\ & + (\epsilon/6u^*)\dot{z}_p(\dot{u}_t\bar{m}_t\dot{X}^{1/3} + \tfrac{1}{2}\dot{z}_{5t}\bar{m}_t^2\dot{X}^{11/3} + \tfrac{1}{3!}\dot{v}_t\bar{m}_t^3\dot{X}^{16/3})\dot{\tau}\ln\dot{\tau}, \end{aligned} \tag{4.46}$$

with

$$\dot{\tau} = 1 + \tfrac{1}{2}\dot{u}_t\bar{m}_t^2\dot{X}^{-1/3} + \tfrac{1}{3!}\dot{z}_{5t}\bar{m}_t\dot{X}^3 + \tfrac{1}{4!}\dot{v}_t\bar{m}_t^4\dot{X}^{14/3}, \tag{4.47}$$

and

$$\bar{m}_t = m|\dot{t}|^{-\beta_t} = m|\dot{t}|^{-1/4}. \tag{4.48}$$

As one might expect, these equations have the same form as (3.51)–(3.53), apart from the additional terms involving $\dot{u}_5$. Of course, (3.51) and (4.46) are two different representations of the equation of state, obtained by solving the inequivalent renormalisation group equations (2.39) and (4.25). They are equal only in the sense that, when expanded to order ϵ, both reproduce (3.29). However, in the plane $u = 0$, we have $H = h$, $M = m$, $\dot{u}_5 = 0$, $\dot{t} = t$ and $X = \dot{X} = 1$. The two equations (3.51) and (4.46) are then identical. Thus the results of this section are complementary to those of § 3 and, taken together, the two expressions for the equation of state yield a consistent description, in scaling form, of crossover from the tricritical point to each of the critical loci.

5. Conclusions

We have used the methods of renormalised perturbation theory to investigate tricritical scaling in the $(4-\epsilon)$-dimensional field theory model defined by the Hamiltonian density

$$\mathcal{H} = \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}t\phi^2 + \tfrac{1}{4!}u\phi^4 + \tfrac{1}{6!}v\phi^6 + \text{counterterms}. \tag{5.1}$$

The interaction $v\phi^6$, which is essential for thermodynamic stability in the region $u \leq 0$, is non-renormalisable in the sense that counterterms proportional to an infinite number of higher-dimensional operators $(\nabla^2\phi)^2$, $\phi^3\nabla^2\phi$, ϕ^8, $\phi^{10}, \ldots$ are required to effect a complete renormalisation of the theory. In practice, however, renormalisation is a perturbative process, which is most systematically performed by considering, at each step, only Feynman graphs containing a given number of loops. In the ϵ expansion, which we have used throughout, critical exponents may be calculated to order ϵ^n by considering only those graphs containing up to n loops. Scaling functions calculated by means of the loopwise expansion may likewise be presented as power series in ϵ, provided that one formally regards v as being of order ϵ^2; in practice this means that the quantity v/u^{*2}, where $u^* = O(\epsilon)$, is treated as being of order unity. At each finite order in the loopwise expansion, only a finite number of operators of higher dimension need be introduced, provided that corrections to scaling due to these operators themselves are neglected. Within these restrictions, a correct treatment of (5.1) involves only a straightforward extension of standard field-theoretic techniques. Our procedure is described in detail in § 2. In fact, the renormalisation of *single* insertions of composite operators such as ϕ^6 is well understood (see e.g. Amit *et al* 1977 and references therein), and at first order in v our procedure is equivalent to the standard theory.

Detailed calculations have been carried out to one-loop order. At this order, considerable simplifications arise, since there is no mixing of ϕ^6 with other dimension-six operators; the only additional operator explicitly required is ϕ^8. For small positive u, we find, as expected, that the equation of state can be written in the form

$$H = |t|^{\Delta_0}\bar{H}(M|t|^{-\beta_0}, u|t|^{-\phi_0}, v|t|^{-\Psi_0}), \tag{5.2}$$

or

$$H = |M|^{\delta_0}\tilde{H}(t|M|^{-1/\beta_0}, u|M|^{-\phi_0/\beta_0}, v|M|^{-\Psi_0/\beta_0}), \tag{5.3}$$

where the Gaussian tricritical exponents Δ_0, β_0, etc are those listed in (1.13). While these equations have the formal appearance of scaling, the Gaussian exponents are in fact misleading as to the actual tricritical behaviour of the system. This is most clearly seen by considering (5.3) in the ordered region of the symmetry plane. On setting $H = u = 0$ with $t < 0$, we find for small t and M that

$$t \approx -v|M|^{1/\beta_t}, \tag{5.4}$$

where

$$\beta_t = \beta_0/(1-\Psi_0) = \tfrac{1}{4}, \tag{5.5}$$

belongs to the set of classical tricritical exponents (1.3). These, therefore, are the exponents which describe the true tricritical behaviour, and should appear in (5.2) and (5.3). Following the spherical model calculation of Sarbach and Fisher (1978a, b), we find that the equation of state can be rewritten in a form involving the physically appropriate classical exponents (1.3), but at the expense of introducing an extra scaling variable p which appears in our formulation as the coefficient of $(\nabla\phi)^2$ in the Hamiltonian

$$\mathcal{H} = \tfrac{1}{2}p^{-2/d}(\nabla\phi)^2 + \ldots. \tag{5.6}$$

Since p can be eliminated by the rescaling of variables, it does not necessitate any further renormalisation. As discussed by Sarbach and Fisher, and in § 3 above, p can be interpreted in terms of the range of pairwise interactions in an underlying lattice model.

In the long-range limit $p \to 0$, spatial fluctuations in ϕ are damped out, and the purely classical theory is regained, as it should be. The equation of state is thus most appropriately written in the form

$$H = |t|^{\Delta_t} \bar{H}_t(M|t|^{-\beta_t}, u|t|^{-\phi_t}, p|t|^{-\phi_p}), \tag{5.7}$$

where

$$\phi_p = -\tfrac{1}{2}(1-\epsilon), \tag{5.8}$$

and v appears simply as a constant. Near the lambda line, our equations may again be rewritten in terms of the usual Ising-like exponents, with corrections appearing in the form

$$(u-u^*)|t|^{-\phi_u}, \qquad v|t|^{-\Psi}, \tag{5.9}$$

where

$$\phi_u = -\tfrac{1}{2}\epsilon + O(\epsilon^2), \qquad \Psi = -(1+\tfrac{5}{3}\epsilon + O(\epsilon^2)). \tag{5.10}$$

For $u \leq 0$, we find two critical loci,

$$t_c = 3u^2/2v, \qquad H_c^{\pm} = \pm(4u^2/5v)(-6u/v)^{1/2} = (4u^2/5v)M_c^{\pm} \tag{5.11}$$

which are the boundaries of two symmetrically disposed first-order surfaces or 'wings'. On shifting the spin field according to $\phi \Rightarrow \phi + M_c^{+}$, we obtain a new, correctly renormalised theory involving the new parameters $\dot{t} = t - t_c$, $\dot{u} = -2u$, and the additional variable

$$\dot{u}_5 = vM_c^{+} \tag{5.12}$$

associated with the operator ϕ^5. Critical behaviour along the wings has the usual Ising-like form, governed by the fixed point $\dot{u} = u^*$, where u^* is the same number as that appearing for $u > 0$, but with additional corrections of the form

$$\dot{u}_5|\dot{t}|^{-\Psi_5}, \tag{5.13}$$

with a negative, irrelevant crossover exponent

$$\Psi_5 = -\tfrac{1}{2}(1 + 2\epsilon + O(\epsilon^2)). \tag{5.14}$$

Crossover to tricritical behaviour is described by an equation analogous to (5.7), with the substitutions $H \to h = H - H_c - \dot{t}M_c$, $M \to M - M_c$, $t \to \dot{t}$, $u \to \dot{u}$, but involving again the additional scaling variable

$$z_{5t} = u_5|\dot{t}|^{-\Psi_{5t}}, \tag{5.15}$$

with

$$\Psi_{5t} = \tfrac{1}{4}. \tag{5.16}$$

The two representations (3.51) and (4.46) for the equation of state, valid respectively when u is positive or negative, are obtained by solving two different renormalisation group equations, and coincide precisely only when expanded to order ϵ. However, they are identical in the plane $u = 0$, and together yield a consistent description of crossover from the tricritical point to each of the critical loci. It should be remarked, however, that we have *not* succeeded in giving a fully complete account of scaling in the tricritical region. The reason is that, in order to locate the fixed point describing one of the wing critical lines for $u > 0$, we wrote the free energy in terms of the magnetisation deviation $m = M - M_c^+$. Now, while the expression (3.27) for the free energy obtained directly from perturbation theory is symmetric about $M = 0$, or $m = -M_c^+$, the scaling form (4.46) for the equation of state obtained in the last section by solving the renormalisation group equation to order ϵ does not have this symmetry. Thus, although our equations describe crossover from the tricritical point to each of the wing critical lines separately, we have not been able to exhibit a single scaling function describing crossover to *both* critical lines. One must, of course, enforce the symmetry by restricting (4.46) to the region $H > 0$ and using the corresponding form with the replacement $m \Rightarrow m' = M - M_c^-$ in the region $H < 0$. However, this yields an unphysical singularity in the equation of state at $H = 0$, and we must conclude that our results are valid only in the immediate vicinity of the critical line.

Finally, we illustrate our results by plotting contours of the susceptibility for small, negative values of u. For this purpose we set $d = 3$ or $\epsilon = 1$. This is actually not strictly appropriate, since one expects to find logarithmic corrections to the classical tricritical behaviour in three dimensions, owing to the marginal nature of ϕ^6 (Stephen *et al* 1975). These corrections are not reproduced by our ϵ expansion technique. Furthermore, we take $\dot{u}$ sufficiently small that factors of $1 - \dot{u}/u^*$ can be set equal to unity. Next, we introduce the rescaled variables

$$H' = H/H_c, \tag{5.17}$$

$$m' = m/M_c, \tag{5.18}$$

$$t' = t/t_c = (\dot{t} + t_c)/t_c. \tag{5.19}$$

After these rescalings, the equation of state (5.7), whose explicit form is given in (4.46), depends, at $\epsilon = 1$, on a single, non-universal parameter

$$z' = pv^{1/2}/u^*. \tag{5.20}$$

A discussion of this non-universal dependence in the spherical model limit has been given by Fisher and Sarbach (1978). For our present illustrative purposes we set $z' = 1$: the limit $z' = 0$ of long-range interactions yields, as discussed above, the mean field theory described in the Introduction.

At this point, the equation of state describes a plane u = constant for small, negative u in the (H, t, u) phase diagram. The bold line in figure 2 represents, for $H' > 0$, the wing coexistence surface, along which the free energy

$$F'(m') = \int_0^{m'} H'(m'')\,\mathrm{d}m'' \tag{5.21}$$

has two equal minima. It terminates at the critical point $(H', t') = (1, 1)$. Also shown in figure 2 are contours of the inverse susceptibility, which is conveniently normalised by

$$\chi^{-1}(t', H') = \tfrac{8}{15}\partial H'/\partial m'. \tag{5.22}$$

In mean field theory this normalisation yields

$$\chi^{-1}_{\mathrm{MFT}}(t', 0) = t'. \tag{5.23}$$

A technical difficulty in computing these contours is that the scaling form (4.46) for the equation of state does not have the required analyticity properties for large values of $y = m/|\dot{t}|^{\beta_t}$. As discussed in § 3, one must use the alternative form

$$h = m^{\delta_t}\tilde{h}(t/|m|^{1/\beta_t}) \tag{5.24}$$

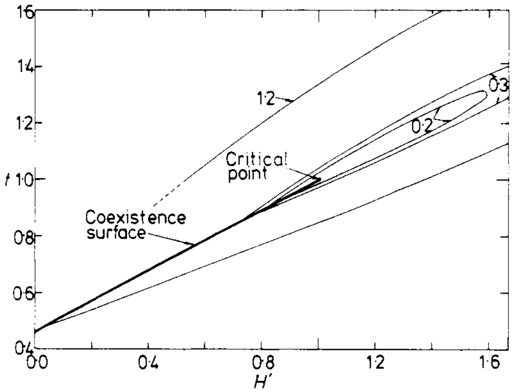

Figure 2. The (H', t') phase diagram for small, negative u and $H' > 0$. The coexistence surface (bold curve) and contours of the inverse susceptibility (light curves) are shown to first order in ϵ, with $\epsilon = 1$.

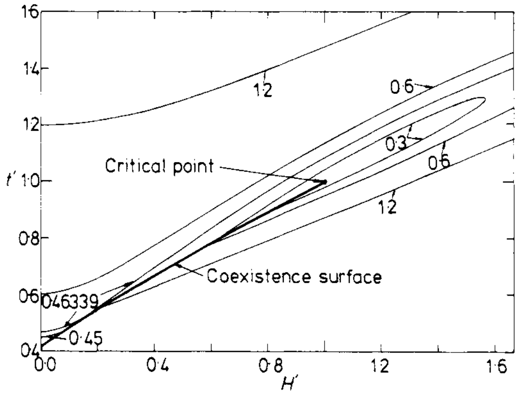

Figure 3. The (H', t') phase diagram for negative u, with contours of the inverse susceptibility, calculated from mean field theory.

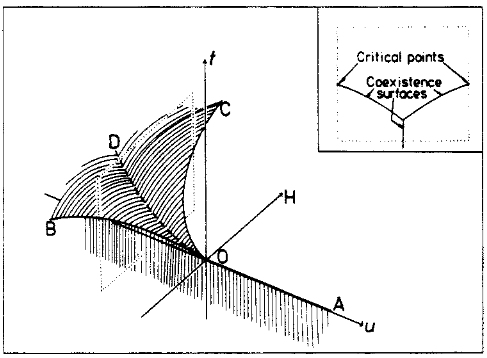

Figure 4. Schematic representation of the tricritical (H, t, u) phase diagram. Shaded areas are the coexistence surfaces, bounded by the critical lines OA (the lambda line), OB and OC, while the broken line OD is a line of triple points. The vertical section indicated by a dotted rectangle is reproduced in the inset, and the right-hand half of this plane appears in figures 2 and 3.

when y is large. In particular, this is necessary to avoid spurious singularities in the susceptibility as $t \to t_c$, with $m \neq 0$. Fortunately, contours evaluated from the appropriate forms of the equation of state in different regions can be smoothly matched and, to avoid confusion, this matching has not been indicated explicitly in figure 2.

As expected, the contours do not behave in the correct manner for small H': they meet the $H' = 0$ axis at a finite angle. In fact, they are probably reliable only near the critical point, and only this region has been mapped out in detail. For comparison, the corresponding diagram for mean field theory is shown in figure 3. Here, by contrast, the contours, which may be obtained analytically, meet the $H' = 0$ axis with zero slope, and the symmetry about this axis is, of course, maintained in a perfectly analytic manner. For orientation, the full tricritical phase diagram is sketched in figure 4.

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